専門科目(TG010001)

流体力学II
Fluid Mechanics II

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１.流体の支配方程式 (Governing Equations)

- 連続の式：

$$\frac{\partial \rho}{\partial t} + \rho div\boldsymbol{V} = 0 \Rightarrow$$

$$\frac{\partial \rho}{\partial t} + \rho(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}) = 0$$ 圧縮性流体

$$\frac{\partial \rho}{\partial t} = 0, (\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}) = 0$$ 非圧縮性流体

- ナビエ・ストークス方程式：

$$\frac{DV}{Dt} = F - \frac{1}{\rho} gradp + \nu \nabla^2 V$$

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}) + F_x$$

$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial y} + \nu(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}) + F_y$$

4. 相似性と無次元化
(Similarity and Nondimensionalization)

- 代表的物理量：
  代表長さ＝L　代表速度＝U　代表時間＝T=L/U
- 無次元化:
  (u,v,w) = U(u*,v*,w*), (x,y,z) = L(x*,y*,z*), t=t*L/U

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) \qquad \mathrm{Re} = \frac{UL}{\nu}$$

$$\frac{U}{L/U}\frac{\partial u^*}{\partial t^*} + \frac{U^2}{L}\left(u^*\frac{\partial u^*}{\partial x^*} + v^*\frac{\partial u^*}{\partial y^*}\right) = -\frac{1}{\rho L}\frac{\partial p}{\partial x^*} + \nu\frac{U}{L^2}\left(\frac{\partial^2 u^*}{\partial x^{*2}} + \frac{\partial^2 u^*}{\partial y^{*2}}\right)$$

$$\frac{\partial u^*}{\partial t^*} + u^*\frac{\partial u^*}{\partial x^*} + v^*\frac{\partial u^*}{\partial y^*} = -\frac{\partial p^*}{\partial x^*} + \frac{1}{UL/\nu}\left(\frac{\partial^2 u^*}{\partial x^{*2}} + \frac{\partial^2 u^*}{\partial y^{*2}}\right) \Leftarrow p^* = \frac{p}{\rho U^2}$$

6. 2次元ポアズイユ流れ１
(2D Poiseuille Flow)

- 静止平行平板間の流れ：
- 支配方程式：

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{\partial p}{\partial x} + \frac{1}{Re}\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) + F_x$$

$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} = -\frac{\partial p}{\partial y} + \frac{1}{Re}\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right) + F_y$$

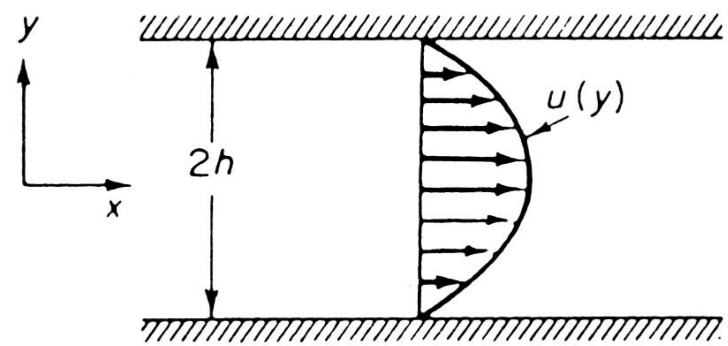
- 問題分析:

定常流れ

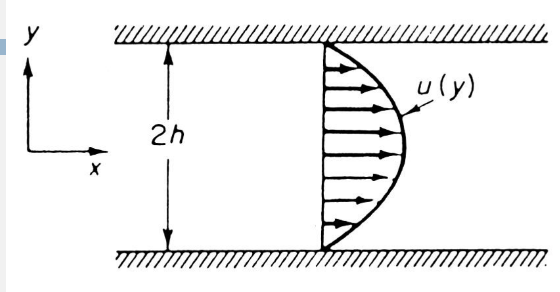
水平方向変化無し

$$\frac{\partial}{\partial t} = 0$$

$$v = 0; \frac{\partial p}{\partial y} = 0; \frac{\partial u}{\partial x} = 0$$



6. 2次元ポアズイユ流れ２
(2D Poiseuille Flow)



- 静止平行平板間の流れ：
- 支配方程式：

$$\frac{\partial u}{\partial x} = 0$$

$$\frac{\partial u}{\partial x} u = -\frac{\partial p}{\partial x} + \frac{1}{Re}\frac{\partial^2 u}{\partial y^2}$$

$$\Rightarrow \quad \frac{dp}{dx} = \frac{1}{Re}\frac{d^2 u}{dy^2}$$

$$u = \frac{Re}{2}\frac{dp}{dx} y^2 + Ay + B$$

- 境界条件：

粘性条件（滑りなし）

$$y = +\frac{b}{2}, u = 0$$

$$y = -\frac{b}{2}, u = 0$$

$$\Rightarrow \quad u = \frac{Re}{2}\frac{dp}{dx}\left(y^2 - \frac{b^2}{4}\right) \Leftarrow \frac{dp}{dx} < 0$$

３次元ハーゲンポアズイユ流れ１ (3D Hagen-Poiseuille Flow)

- 真直ぐ円管内の流れ：
- 支配方程式（円柱座標へ）：

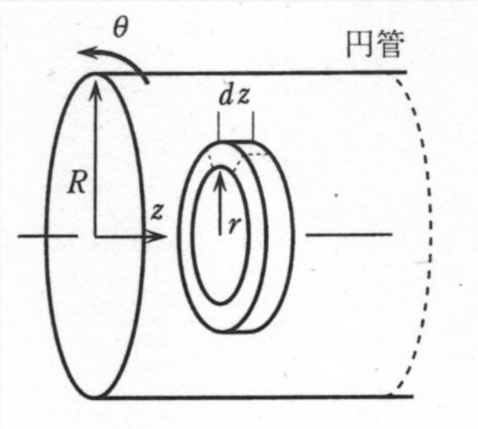
$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}=0 \qquad \frac{DV}{Dt}=F-\frac{1}{\rho}gradp+\nu\nabla^2 V$$

$$x=r\cos\theta, y=r\sin\theta, z=z$$

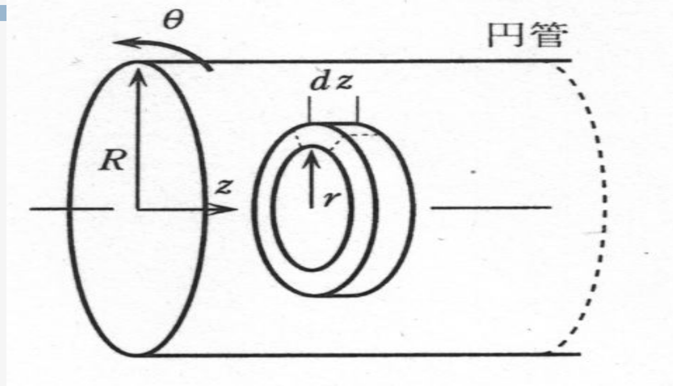
$$u=\frac{\partial x}{\partial t}=\frac{\partial r}{\partial t}\cos\theta-r\sin\theta\frac{\partial\theta}{\partial t}=v_r\cos\theta-r\sin\theta v_\theta$$

$$v=\frac{\partial y}{\partial t}=\frac{\partial r}{\partial t}\sin\theta+r\cos\theta\frac{\partial\theta}{\partial t}=v_r\sin\theta+r\cos\theta v_\theta$$

$$w=\frac{\partial z}{\partial t}$$



３次元ハーゲンポアズイユ流れ2
(3D Hagen-Poiseuille Flow)



- **支配方程式（円柱座標へ）：**

$$\frac{1}{r}\frac{\partial(rv_r)}{\partial r}+\frac{1}{r}\frac{\partial v_\theta}{\partial \theta}+\frac{\partial v_z}{\partial z}=0$$

$$\frac{\partial v_z}{\partial t}+(V\bullet\nabla)v_z=-\frac{\partial p}{\partial z}+\frac{1}{\boldsymbol{Re}}(\nabla^2 v_z)$$

$$\frac{\partial v_r}{\partial t}+(V\bullet\nabla)v_r-\frac{v_\theta^2}{r}=-\frac{\partial p}{\partial r}+\frac{1}{\boldsymbol{Re}}(\nabla^2 v_r-\frac{v_r}{r^2}-\frac{2}{r^2}\frac{\partial v_\theta}{\partial \theta})$$

$$\frac{\partial v_\theta}{\partial t}+(V\bullet\nabla)v_\theta+\frac{v_r v_\theta}{r}=-\frac{1}{r}\frac{\partial p}{\partial \theta}+\frac{1}{\boldsymbol{Re}}(\nabla^2 v_\theta-\frac{v_\theta}{r^2}+\frac{2}{r^2}\frac{\partial v_r}{\partial \theta})$$

- **問題分析:**

定常流れ　$\frac{\partial}{\partial t}=0$

水平方向変化無し　$v_r=v_\theta=0;$

$$\frac{\partial p}{\partial r}=\frac{\partial p}{\partial \theta}=0;\frac{\partial w}{\partial z}=0$$

３次元ハーゲンポアズイユ流れ3
(3D Hagen-Poiseuille Flow)

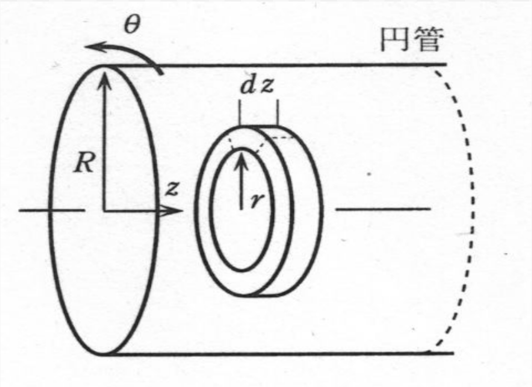
- **支配方程式：**

$$\frac{\partial w}{\partial z} = 0$$

$$0 = -\frac{dp}{dz} + \frac{1}{Re}\left(\frac{d^2 w}{dr^2} + \frac{1}{r}\frac{dw}{dr}\right)$$

$$\frac{dp}{dz} = \frac{1}{Re}\left(\frac{d^2 w}{dr^2} + \frac{1}{r}\frac{dw}{dr}\right)$$

$$w = \frac{\boldsymbol{Re}}{4}\frac{dp}{dz}r^2 + A\,\mathbf{ln}\,r + B$$



- **境界条件：**

粘性条件（滑りなし）

$$r = R, w = 0;$$

$$r = 0, w \to \infty, A = 0$$

$$w = \frac{Re}{4}\frac{dp}{dz}(r^2 - R^2), \frac{dp}{dz} < 0$$

$$w = \frac{1}{4\mu}\frac{dp}{dz}(r^2 - R^2)$$

３次元ハーゲンポアズイユ流れ4 (3D Hagen-Poiseuille Flow)

- **流量、断面平均流速、摩擦応力と係数：**

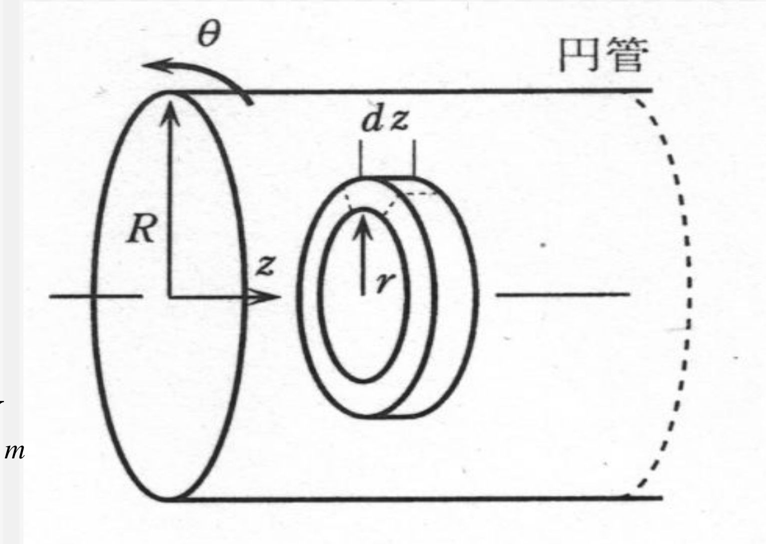
$$Q = \int_0^R w \bullet 2\pi r dr = \int_0^R \frac{2\pi r}{4\mu}\frac{dp}{dz}(r^2 - R^2)dr = -\frac{\pi}{8\mu}\frac{dp}{dz}R^4$$

$$U_m = \frac{Q}{\pi R^2} = -\frac{1}{8\mu}\frac{dp}{dz}R^2$$

$$Re = \frac{\rho U_m (2R)}{\mu}$$

$$\tau_0 = \mu\frac{dw}{dy} = -\mu\frac{dw}{dr}\Big|_{r=R} = -\frac{1}{2}\frac{dp}{dz}R = \frac{4\mu}{R}U_m$$

$$C_f = \frac{\tau_0}{0.5\rho U_m^2} = \frac{8\mu}{\rho R U_m} = \frac{16}{Re}$$



レイリー流れ１
(2D Raily Flow)

- 平板運動による非定常流れ（スタート流れ）：
- 支配方程式：

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0$$

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=-\frac{\partial p}{\partial x}+\frac{1}{Re}\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right)+F_x$$

$$\frac{\partial v}{\partial t}+u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}=-\frac{\partial p}{\partial y}+\frac{1}{Re}\left(\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}\right)+F_y$$

- 問題分析:

非定常流れ
水平方向変化無し

$$\frac{\partial v}{\partial y}=0;\frac{\partial p}{\partial x}=0;\frac{\partial u}{\partial x}=0$$

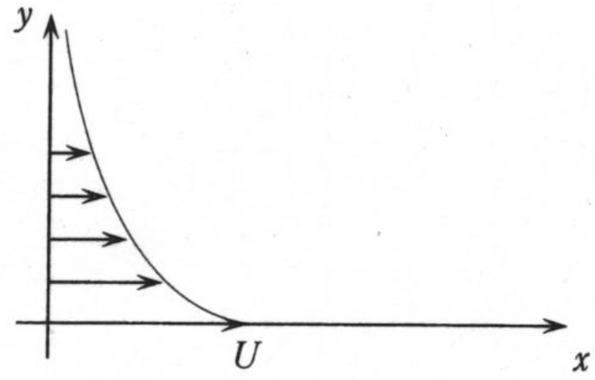


図 5·10　レイリー問題

レイリー流れ2
(2D Raily Flow)

- **平板運動による非定常流れ（スタート流れ）：**
- **支配方程式：** **無次元座標の導入へ**

$$v = 0$$

$$\frac{\partial p}{\partial x} = 0$$

$$\frac{\partial u}{\partial t} = \nu \frac{\partial^2 u}{\partial y^2}$$

$$\delta \propto \sqrt{\nu t} \Rightarrow \eta = \frac{y}{2\sqrt{\nu t}} ; g(\eta) = \frac{u(y,t)}{U}$$

$$\frac{d^2 g}{d\eta^2} + 2\eta \frac{dg}{d\eta} = 0 \Rightarrow \log \frac{dg}{d\eta} = -\eta^2 + \log C$$

$$\frac{dg}{d\eta} = C exp(-\eta^2)$$

- **初期条件・境界条件：**

$$IC : t \le 0, y \ge 0 \Rightarrow u = 0$$

$$BC : t > 0, y = 0 \Rightarrow u = U;$$

$$t > 0, y \ge 1 \Rightarrow u \to 0$$

$$BC : \eta = 0 \Rightarrow g = 1;$$

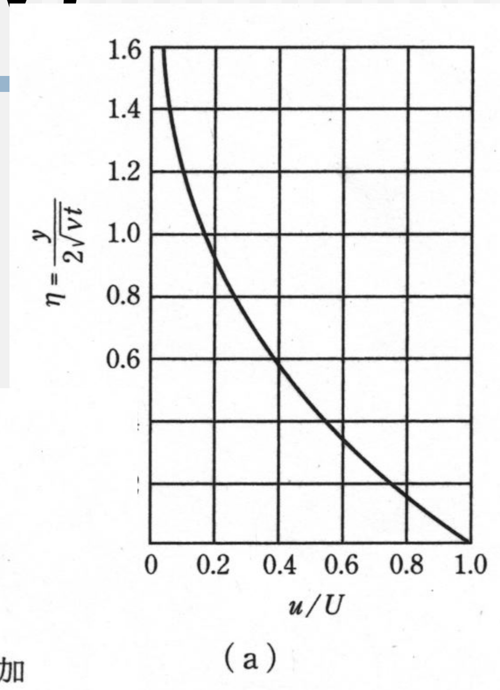
$$\eta \ge 1 \Rightarrow g \to 0$$

レイリー流れ3
(2D Raily Flow)

- 平板運動による非定常流れ（スタート流れ）
- 支配方程式：　　　　**無次元座標の導入へ**

$$\frac{dg}{d\eta} = C\,\boldsymbol{exp}(-\eta^2) \Rightarrow g(\eta) = \frac{u}{U} = 1 - \frac{2}{\sqrt{\pi}}\int_0^{\eta} e^{-\eta^2} d\eta$$

$$where \int_0^{\infty} e^{-\eta^2} d\eta = \frac{\sqrt{\pi}}{2}$$



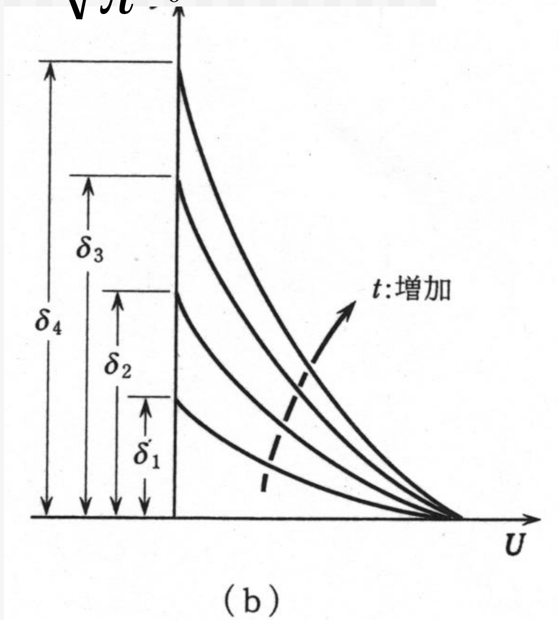


図 5•12　レイリー問題の速度分布

- 相似座標:
- 粘性層:
- 熱伝導:

$$\eta = 2 \Rightarrow \delta = 4\sqrt{\nu t}$$

振動平板流れ1
(Flow with an Oscillating Plate)

- 平板振動による非定常流
- 支配方程式：

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0$$

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=-\frac{\partial p}{\partial x}+\frac{1}{Re}\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right)+F_x$$

$$\frac{\partial v}{\partial t}+u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}=-\frac{\partial p}{\partial y}+\frac{1}{Re}\left(\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}\right)+F_y$$

- 問題分析：

非定常流れ
水平方向変化無し

$$\frac{\partial v}{\partial y}=0;\frac{\partial p}{\partial x}=0;\frac{\partial u}{\partial x}=0$$

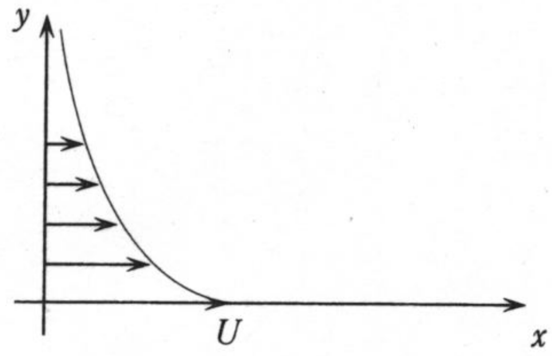


図 5・10　レイリー問題

振動平板流れ２ (Flow with an Oscillating Plate)

- 平板振動による非定常流
- 支配方程式：

$$v = 0$$

$$\frac{\partial p}{\partial x} = 0$$

$$\frac{\partial u}{\partial t} = \nu \frac{\partial^2 u}{\partial y^2}$$

$$u = f(y)e^{i\omega t}$$

$$where, u = Ue^{i\omega t} = U(\cos \omega t + i \sin \omega t)$$

$$\frac{d^2 f}{dy^2} = \frac{i\omega f}{\nu}, General Solution: f = A\exp(\alpha y)$$

- 初期条件と境界条件：

非定常流れ

水平方向変化無し

$$BC: y = 0 \Rightarrow u = U\cos \omega t;$$

$$t > 0, y \geq 1 \Rightarrow u \to 0$$

振動平板流れ3
(Flow with an Oscillating Plate)

- **支配方程式：**

$$\frac{d^2 f}{dy^2} = \frac{i\omega f}{\nu}, General Solution: f = A\,\mathbf{exp}(\alpha y)$$

$$u = Ue^{-ky}\,\mathbf{cos}(\omega t - ky)$$

$$where, A = U;$$

$$\alpha = -k - ik, k = \sqrt{\omega/(2\nu)}$$

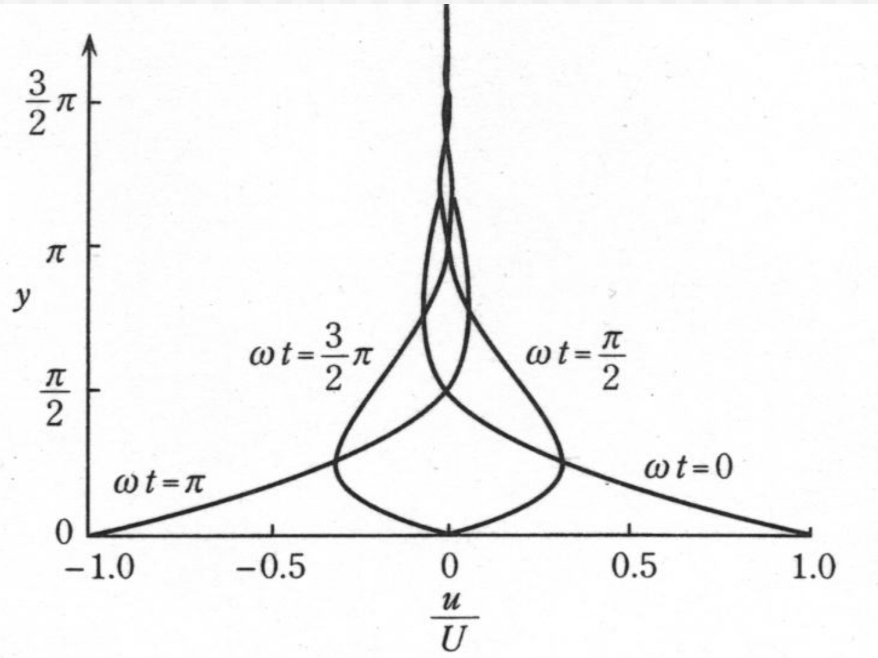
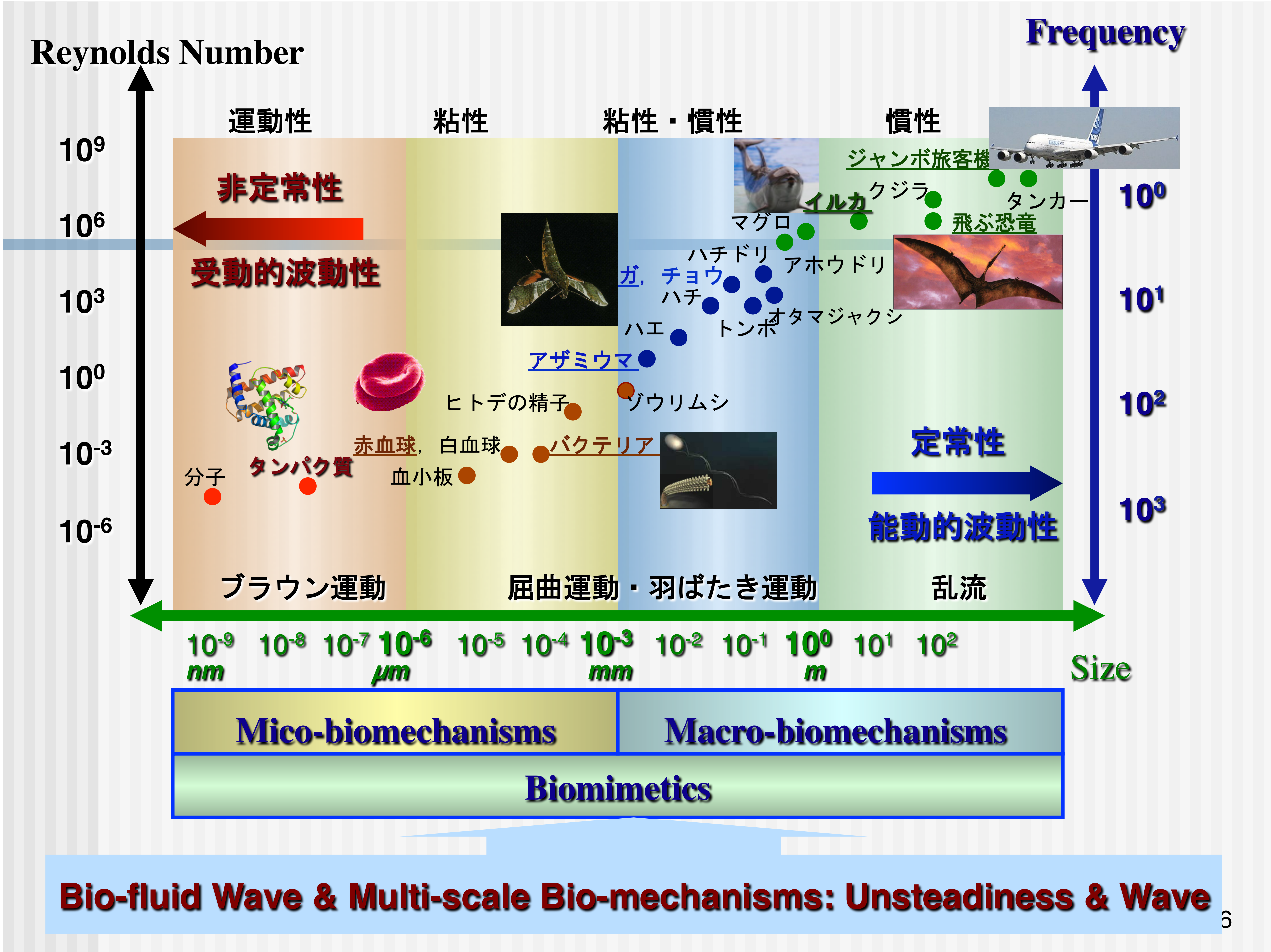


図 5·13 振動平板による流れ



Reynolds Number
Frequency
運動性
粘性
粘性・慣性
慣性
10⁹
10⁶
10³
10⁰
10⁻³
10⁻⁶
非定常性
受動的波動性
ジャンボ旅客機
クジラ
タンカー
イルカ
マグロ
飛ぶ恐竜
ハチドリ
アホウドリ
ガ，チョウ
ハチ
オタマジャクシ
ハエ
トンボ
アザミウマ
ゾウリムシ
ヒトデの精子
赤血球，白血球
バクテリア
血小板
分子
タンパク質
定常性
能動的波動性
10⁰
10¹
10²
10³
ブラウン運動
屈曲運動・羽ばたき運動
乱流
10⁻⁹ nm
10⁻⁸
10⁻⁷
10⁻⁶ μm
10⁻⁵
10⁻⁴
10⁻³ mm
10⁻²
10⁻¹
10⁰ m
10¹
10²
Size
Mico-biomechanisms
Macro-biomechanisms
Biomimetics
Bio-fluid Wave & Multi-scale Bio-mechanisms: Unsteadiness & Wave

ストークス流れ1
(Stokesian Flow)

- 支配方程式：

$$divV = 0 \qquad \frac{\partial V}{\partial t} + V \quad V = F - \frac{1}{\rho} gradp + \nu \nabla^2 V$$

- ストークス近似：移流項を無視
- 一様流れの中に置かれた球まわりの流れ：

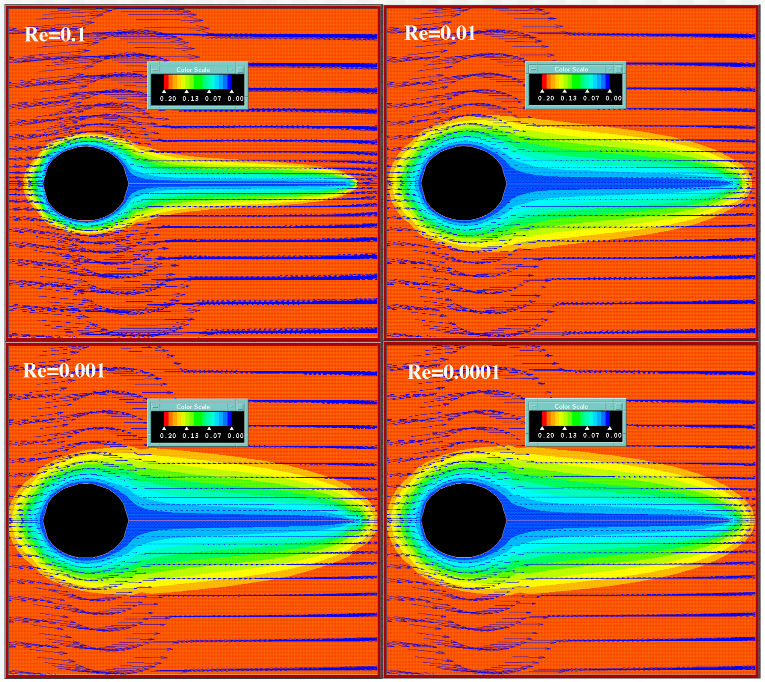
$BodySurface : u = v = w = 0$

$OutsideBoundary : u = U , p = p_\infty$

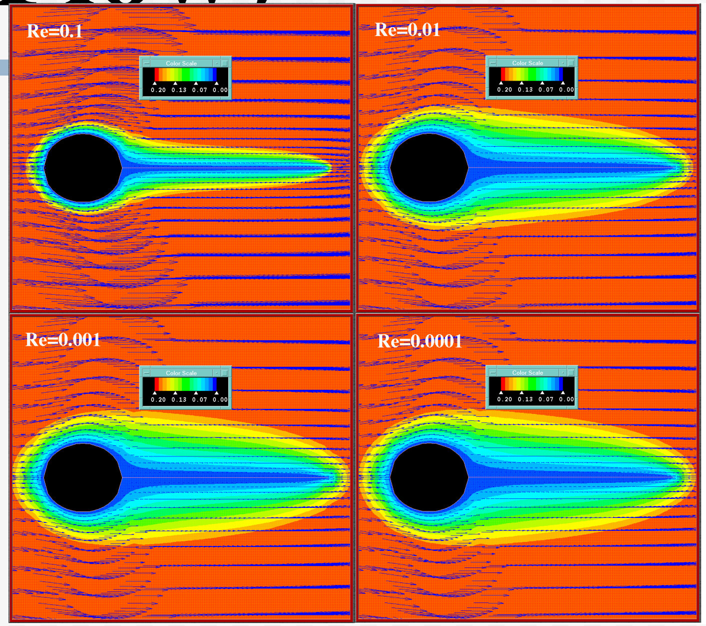
$$v_r = U \cos\theta \left(1 - \frac{3/2 \bullet a}{r} + \frac{1/2 \bullet a^3}{r^3}\right)$$

$$v_\theta = -U \sin\theta \left(1 - \frac{3/4 \bullet a}{r} - \frac{1/4 \bullet a^3}{r^3}\right)$$

$$p = p_\infty - \frac{3}{2}\mu \frac{U\cos\theta}{a}, \tau_{r\theta} = \frac{3}{2}\mu\frac{U\sin\theta}{a}$$



ストークス流れ2 (Stokesian Flow)



- 抵抗の式：

$$Drag_x = -\int_0^{\pi} \tau_{r\theta}\, \mathbf{sin}\,\theta ds - \int_0^{\pi} p\, \mathbf{cos}\,\theta ds$$

$$= 4\pi a\mu U + 2\pi a\mu U = 6\pi a\mu U$$

$$C_d = \frac{Drag}{\frac{1}{2}\rho U^2 \pi a^2} = \frac{24}{\boldsymbol{Re}}$$

- ２次元円柱まわりのストークス流れが存在しない（数学的証明）
ストークスのパラドックス

＠オゼーン近似：

$$\frac{\partial V}{\partial t} + U\frac{\partial V}{\partial x} = F - \frac{1}{\rho} gradp + \nu \nabla^2 V$$

$$Drag_x = -\int_0^{\pi} \tau_{r\theta}\, \mathbf{sin}\,\theta ds - \int_0^{\pi} p\, \mathbf{cos}\,\theta ds$$

$$C_d = \frac{Drag}{\frac{1}{2}\rho U^2 \pi a^2} = \frac{24}{\boldsymbol{Re}}\left(1 + \frac{3}{16}\boldsymbol{Re}\right)$$